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Vector norm inequalities for power series of operators in Hilbert spaces

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Abstract

In this paper, vector norm inequalities that provides upper bounds for the Lipschitz quantity $\|f(T)x - f(V)x\|$ for power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, bounded linear operators T, V on the Hilbert space H and vectors $x \in H$ are established. Applications in relation to Hermite-Hadamard type inequalities and examples for elementary functions of interest are given as well.

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1 Introduction

Associated to a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have naturally another power series with coefficients being the absolute values of those of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is well known that this two power series have the same radius of convergence. Observe that we trivially have $f_a = f$ if all coefficients $a_n \geq 0$.

We notice that if

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z}, \; z \in D(0,1); \tag{1.1}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \; z \in \mathbb{C};$$

$$h(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \; z \in \mathbb{C};$$

$$l(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \; z \in D(0,1);$$

where $D(0,1)$ is the open disk centered in 0 and of radius 1, then the corresponding functions

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constructed by the use of the absolute values of the coefficients are

$$f_a(z) = \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \ln \frac{1}{1-z}, \ z \in D(0,1); \tag{1.2}$$
$$g_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \ z \in \mathbb{C};$$
$$h_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \ z \in \mathbb{C};$$
$$l_a(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ z \in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \ z \in \mathbb{C}; \tag{1.3}$$
$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1);$$
$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1) n!} z^{2n+1}, \ z \in D(0,1);$$
$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1);$$
$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0,$$
$$z \in D(0,1);$$

where Γ is *Gamma function*.

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a separable complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known [3] that in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\| |A| - |B| \| \le L \|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [11], [12] and Kato in [17], the following inequality holds

$$\| |A| - |B| \| \le \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.4}$$

for any $A, B \in \mathcal{B}(H)$ with $A \ne B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (trC^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \leq \sqrt{2}\,\|A - B\|_{HS} \tag{1.5}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be self-adjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right), \tag{1.6}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.7}$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \geq aI_H > 0$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [13] and the references therein.

We recall the following result that provides a quasi-Lipschitzian condition for functions defined by power series [9]:

Theorem 1. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R), R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then

$$\|f(T) - f(V)\| \leq f_a'(\max\{\|T\|, \|V\|\}) \|T - V\|. \tag{1.8}$$

If $\|T\|, \|V\| \leq M < R$, then from (1.8) we have the simpler inequality

$$\|f(T) - f(V)\| \leq f_a'(M) \|T - V\| \tag{1.9}$$

We define the *absolute value* of an operator $A \in \mathcal{B}(H)$ defined as $|A|$ as the *square root operator* of the positive operator A^*A. With this notation, we have:

Corollary 1. With the above assumptions for f, we have

$$\|f(T) - f(T^*)\| \leq f_a'(\|T\|) \|T - T^*\| \tag{1.10}$$

if $T \in \mathcal{B}(H)$ with $\|T\| < R$ and

$$\left\|f\left(|N^*|^2\right) - f\left(|N|^2\right)\right\| \leq f_a'\left(\|N\|^2\right) \left\||N^*|^2 - |N|^2\right\| \tag{1.11}$$

if $N \in \mathcal{B}(H)$ with $\|N\|^2 < R$.

Remark 1. With the assumption of Theorem 1 we have

$$\|f(|T|) - f(|V|)\| \le f_a'(\max\{\|T\|, \|V\|\}) \||T| - |V|\|$$

provided $\|T\|, \|V\| < R$.

Motivated by the above results, in this paper we establish some upper bounds for the vector norms

$$\|f(T)x - f(V)x\|, \quad \left\|f\left(\frac{U+V}{2}\right)x - \int_0^1 f((1-s)U + sV)\,xds\right\|$$

and

$$\left\|\frac{f(U)x + f(V)x}{2} - \int_0^1 f((1-s)U + tV)\,xds\right\|$$

where $x \in H$, for various assumptions on the power series $f(z) := \sum_{n=0}^{\infty} a_n z^n$ and the bounded linear operators $T, V \in \mathcal{B}(H)$. Applications for some elementary functions of interest are also provided.

2 Vector Inequalities

The following result also holds:

Theorem 2. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R), R > 0$. If $T, V \in \mathcal{B}(H)$ are commutative and such that $\|T\|, \|V\| < R$, then

$$\|f(T)x - f(V)x\| \le f_a'(\max\{\|T\|, \|V\|\}) \|Tx - Vx\| \tag{2.1}$$

for any $x \in H$.

Proof. We show first that the following power inequality holds true for any $n \in \mathbb{N}$

$$\|T^n x - V^n x\| \le n(\max\{\|T\|, \|V\|\})^{n-1} \|Tx - Vx\| \tag{2.2}$$

for any $x \in H$.

We prove this by induction. We observe that for $n = 0$ and $n = 1$ the inequality reduces to an equality.

Assume now that (2.2) is true for $k \in \mathbb{N}$, $k \ge 1$ and let us prove it for $k + 1$.

Utilising the properties of the operator norm, we have

$$\begin{aligned}\left\|T^{k+1}x - V^{k+1}x\right\| &= \left\|T^k(T - V)x + \left(T^k - V^k\right)Vx\right\| \\ &\le \left\|T^k(T - V)x\right\| + \left\|\left(T^k - V^k\right)Vx\right\| =: I\end{aligned}$$

Since T and V are commutative, then $T^k - V^k$ and V are commutative and

$$I = \left\|T^k(T - V)x\right\| + \left\|V\left(T^k - V^k\right)x\right\|.$$

By the induction hypothesis we have

$$\begin{aligned}
I &\leq \left\|T^k\right\| \|Tx - Vx\| + \|V\| \left\|T^k x - V^k x\right\| \\
&\leq \|T\|^k \|Tx - Vx\| + k \left(\max\{\|T\|, \|V\|\}\right)^{k-1} \|Tx - Vx\| \|V\| \\
&\leq \max\left\{\|T\|^k, \|V\|^k\right\} \|Tx - Vx\| \\
&+ k \left(\max\{\|T\|, \|V\|\}\right)^{k-1} \|Tx - Vx\| \max\{\|T\|, \|V\|\} \\
&= \left(\max\{\|T\|, \|V\|\}\right)^k \|Tx - Vx\| \\
&+ k \left(\max\{\|T\|, \|V\|\}\right)^k \|Tx - Vx\| \\
&= (k+1) \left(\max\{\|T\|, \|V\|\}\right)^k \|Tx - Vx\|
\end{aligned}$$

for any $x \in H$ and the inequality (2.2) is proved.

Now, for any $m \geq 1$, by making use of the inequality (2.2) we have

$$\begin{aligned}
\left\|\sum_{n=0}^{m} a_n T^n x - \sum_{n=0}^{m} a_n V^n x\right\| &\leq \sum_{n=0}^{m} |a_n| \|T^n x - V^n x\| \\
&\leq \|Tx - Vx\| \sum_{n=0}^{m} n |a_n| \left(\max\{\|T\|, \|V\|\}\right)^{n-1}
\end{aligned} \tag{2.3}$$

for any $x \in H$.

Since the series $\sum_{n=0}^{\infty} a_n T^n x, \sum_{n=0}^{\infty} a_n V^n x$ and $\sum_{n=0}^{\infty} n |a_n| \left(\max\{\|T\|, \|V\|\}\right)^{n-1}$ are convergent for any $x \in H$, then by letting $m \to \infty$ in (2.3) we get the inequality (2.1). ∎

Remark 2. If we assume that $\|T\|, \|V\| \leq M < R$, then from (2.1) we can get the simpler inequality

$$\|f(T)x - f(V)x\| \leq f_a'(M) \|Tx - Vx\| \tag{2.4}$$

for any $x \in H$.

Corollary 2. With the assumptions from Theorem 2 for f, we have

$$\|f(N)x - f(N^*)x\| \leq f_a'(\|N\|) \|Nx - N^*x\| \tag{2.5}$$

for any $x \in H$, if $N \in \mathcal{B}(H)$ is a normal operator with $\|N\| < R$.

Since N is normal, then N commutes with N^* and by applying (2.1) for $T = N$ and $V = N^*$ we get (2.5).

Now, if we take $f(z) = \exp z$, $z \in \mathbb{C}$, then we get from (2.1)

$$\|\exp(T)x - \exp(V)x\| \leq \exp\left(\max\{\|T\|, \|V\|\}\right) \|Tx - Vx\| \tag{2.6}$$

for any $x \in H$ and $T, V \in \mathcal{B}(H)$ commuting operators.

If we take $f(z) = \sinh z$, $z \in \mathbb{C}$ and $f(z) = \sin z$, $z \in \mathbb{C}$, then we get from (2.1)

$$\begin{aligned}
&\max\{\|\sinh(T)x - \sinh(V)x\|, \|\sin(T)x - \sin(V)x\|\} \\
&\leq \cosh\left(\max\{\|T\|, \|V\|\}\right) \|Tx - Vx\|
\end{aligned} \tag{2.7}$$

for any $x \in H$ and $T, V \in \mathcal{B}(H)$ commuting operators.

If we consider the function $f(z) = (1 \pm z)^{-1}, z \in D(0,1)$, then we get from (2.1)

$$\left\| (1_H \pm T)^{-1} x - (1_H \pm V)^{-1} x \right\| \le \frac{1}{(1 - \max\{\|T\|, \|V\|\})^2} \|Tx - Vx\| \tag{2.8}$$

for any $x \in H$ and $T, V \in \mathcal{B}(H)$ commuting operators with $\|T\|, \|V\| < 1$.

Now, if we drop the commutativity assumption for the operators involved, we can prove the following result as well:

Theorem 3. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R), R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then

$$\begin{aligned} &\|f(\|Tx\|) Tx - f(\|Vx\|) Vx\| \\ &\le [f_a(\max\{\|Tx\|, \|Vx\|\}) + \max\{\|Tx\|, \|Vx\|\} f_a'(\max\{\|Tx\|, \|Vx\|\})] \\ &\times \|Tx - Vx\| \end{aligned} \tag{2.9}$$

for any $x \in H$, $\|x\| \le 1$.

If $R = \infty$, then the inequality (2.9) holds for any $x \in H$.

Proof. We show first that the following power inequality holds true for any $n \in \mathbb{N}$ and $x \in H$

$$\| \|Tx\|^n Tx - \|Vx\|^n Vx \| \le (n+1) (\max\{\|Tx\|, \|Vx\|\})^n \|Tx - Vx\|. \tag{2.10}$$

For $n = 0$, the inequality becomes an equality.

Assume that $n \ge 1$, then we have

$$\begin{aligned} &\| \|Tx\|^n Tx - \|Vx\|^n Vx \| \\ &= \| \|Tx\|^n Tx - \|Tx\|^n Vx + \|Tx\|^n Vx - \|Vx\|^n Vx \| \\ &\le \| \|Tx\|^n (Tx - Vx) \| + \| (\|Tx\|^n - \|Vx\|^n) Vx \| \\ &= \|Tx\|^n \|Tx - Vx\| + | \|Tx\|^n - \|Vx\|^n | \|Vx\| \\ &\le (\max\{\|Tx\|, \|Vx\|\})^n \|Tx - Vx\| \\ &+ | \|Tx\|^n - \|Vx\|^n | \max\{\|Tx\|, \|Vx\|\}. \end{aligned} \tag{2.11}$$

On the other hand

$$\begin{aligned} | \|Tx\|^n - \|Vx\|^n | &= | \|Tx\| - \|Vx\| | \left(\|Tx\|^{n-1} + ... + \|Vx\|^{n-1} \right) \\ &\le n \|Tx - Vx\| (\max\{\|Tx\|, \|Vx\|\})^{n-1}. \end{aligned} \tag{2.12}$$

Using (2.11) and (2.12) we have

$$\begin{aligned} \| \|Tx\|^n Tx - \|Vx\|^n Vx \| &\le (\max\{\|Tx\|, \|Vx\|\})^n \|Tx - Vx\| \\ &+ n \|Tx - Vx\| (\max\{\|Tx\|, \|Vx\|\})^n \\ &= (n+1) (\max\{\|Tx\|, \|Vx\|\})^n \|Tx - Vx\| \end{aligned}$$

and the inequality (2.10) is proved.

Now, for any $m \geq 1$, by making use of the inequality (2.10) we have

$$\left\| \left(\sum_{n=0}^{m} a_n \|Tx\|^n \right) Tx - \left(\sum_{n=0}^{m} a_n \|Vx\|^n \right) Vx \right\| \tag{2.13}$$
$$\leq \sum_{n=0}^{m} |a_n| \left\| \|Tx\|^n Tx - \|Vx\|^n Vx \right\|$$
$$\leq \|Tx - Vx\| \sum_{n=0}^{m} (n+1) |a_n| \left(\max \{\|Tx\|, \|Vx\|\} \right)^n$$
$$= \|Tx - Vx\| \left(\sum_{n=0}^{m} |a_n| \left(\max \{\|Tx\|, \|Vx\|\} \right)^n + \sum_{n=0}^{m} n |a_n| \left(\max \{\|Tx\|, \|Vx\|\} \right)^n \right)$$
$$= \|Tx - Vx\| \left(\sum_{n=0}^{m} |a_n| \left(\max \{\|Tx\|, \|Vx\|\} \right)^n + \sum_{n=1}^{m} n |a_n| \left(\max \{\|Tx\|, \|Vx\|\} \right)^n \right).$$

Since $\|T\|, \|V\| < R$ and $\|x\| \leq 1$, then the following series are convergent and

$$\sum_{n=0}^{\infty} a_n \|Tx\|^n = f(\|Tx\|), \quad \sum_{n=0}^{\infty} a_n \|Vx\|^n = f(\|Vx\|),$$

$$\sum_{n=0}^{\infty} |a_n| \left(\max \{\|Tx\|, \|Vx\|\} \right)^n = f_a \left(\max \{\|Tx\|, \|Vx\|\} \right)$$

and

$$\sum_{n=1}^{\infty} n |a_n| \left(\max \{\|Tx\|, \|Vx\|\} \right)^n = \max \{\|Tx\|, \|Vx\|\} f_a' \left(\max \{\|Tx\|, \|Vx\|\} \right),$$

then by letting $m \to \infty$ in (2.13) we deduce the desired result (2.9).

If $R = \infty$, then the above series are convergent for any $x \in H$. ∎

Remark 3. A similar result may be proved if one assumes the slightly more general condition that $T, V \in \mathcal{B}(H)$ and $x \in H$ are such that $\|Tx\|, \|Vx\| < R$.

By taking various elementary functions, one can get some examples similar to those above. However, the details are omitted.

3 Applications for Hermite-Hadamard Type Inequalities

The following result is well known in the Theory of Inequalities as the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}$$

for any convex function $f:[a,b]\to\mathbb{R}$.

The distance between the middle and the left term for Lipschitzian functions with the constant $L>0$ has been estimated in [7] to be

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{4}L(b-a) \tag{3.1}$$

while the distance between the right term and the middle term satisfies the inequality [21]

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4}L(b-a). \tag{3.2}$$

For other Hermite-Hadamard type inequalities, see [6], [8], [14], [15], [16], [18], [20], [21], [23], [24], [25], [26] and [27].

In order to extend these results to functions of operators we need the following lemma that is of interest in itself as well:

Lemma 1. Let $f:\mathcal{C}\subset\mathcal{B}(H)\to\mathcal{B}(H)$ be a vector L-Lipschitzian function on the convex set $\mathcal{C}$, i.e. it satisfies

$$\|f(U)x - f(V)x\| \le L\|Ux - Vx\| \text{ for any } U,V\in\mathcal{C} \text{ and } x\in H.$$

For $U,V\in\mathcal{C}$ and $x\in H\setminus\{0\}$, define the function $\varphi_{U,V,x}:[0,1]\to H$ by

$$\begin{aligned}\varphi_{U,V,x}(t) &:= \frac{1}{2}\left[f\left((1-t)U + t\frac{U+V}{2}\right)x + f\left(t\frac{U+V}{2} + (1-t)V\right)x\right] \\ &= \frac{1}{2}\left[f\left(\left(1-\frac{t}{2}\right)U + \frac{t}{2}V\right)x + f\left(\frac{t}{2}U + \left(1-\frac{t}{2}\right)V\right)x\right].\end{aligned}$$

Then for any $t_1,t_2\in[0,1]$ we have the inequality

$$\|\varphi_{U,V,x}(t_2) - \varphi_{U,V,x}(t_1)\| \le \frac{1}{2}L\|Ux - Vx\|\,|t_2 - t_1|, \tag{3.3}$$

i.e., the function $\varphi_{U,V,x}$ is Lipschitzian with the constant $\frac{1}{2}L\|Ux - Vx\|$.

In particular, we have the inequalities

$$\left\|f\left(\frac{U+V}{2}\right)x - \varphi_{U,V,x}(t)\right\| \le \frac{1}{2}L\|Ux - Vx\|(1-t), \tag{3.4}$$

$$\left\|\frac{f(U)x + f(V)x}{2} - \varphi_{U,V,x}(t)\right\| \le \frac{1}{2}L\|Ux - Vx\|\,t \tag{3.5}$$

and

$$\left\| \frac{1}{2}\left[f\left(\frac{3U+V}{2}\right) x+f\left(\frac{U+3V}{2}\right) x\right] -\varphi_{U,V,x}\left(t\right) \right\| \le \frac{1}{2}L\left\| Ux-Vx\right\| \left| t-\frac{1}{2}\right| \tag{3.6}$$

for any $t\in [0,1]$.

Proof. We have

$$\begin{aligned}
&\left\| \varphi_{U,V,x}\left(t_2\right) -\varphi_{U,V,x}\left(t_1\right) \right\| \\
&=\frac{1}{2}\left\| f\left(\left(1-t_2\right) U+t_2\frac{U+V}{2}\right) x+f\left(t_2\frac{U+V}{2}+\left(1-t_2\right) V\right) x \right.\\
&\left. -f\left(\left(1-t_1\right) U+t_1\frac{U+V}{2}\right) x-f\left(t_1\frac{U+V}{2}+\left(1-t_1\right) V\right) x\right\| \\
&\le \frac{1}{2}\left\| f\left(\left(1-t_2\right) U+t_2\frac{U+V}{2}\right) x-f\left(\left(1-t_1\right) U+t_1\frac{U+V}{2}\right) x\right\| \\
&+\frac{1}{2}\left\| f\left(t_2\frac{U+V}{2}+\left(1-t_2\right) V\right) x-f\left(\left(1-t_1\right) U+t_1\frac{U+V}{2}\right) x\right\| \\
&\le \frac{1}{2}L\left\| \left(1-t_2\right) Ux+t_2\frac{Ux+Vx}{2}-\left(1-t_1\right) Ux-t_1\frac{Ux+Vx}{2}\right\| \\
&+\frac{1}{2}L\left\| t_2\frac{Ux+Vx}{2}+\left(1-t_2\right) Vx-\left(1-t_1\right) Ux-t_1\frac{Ux+Vx}{2}\right\| \\
&=\frac{1}{4}L\left\| Ux-Vx\right\| \left| t_2-t_1\right| +\frac{1}{4}L\left\| Ux-Vx\right\| \left| t_2-t_1\right| =\frac{1}{2}L\left\| Ux-Vx\right\| \left| t_2-t_1\right|
\end{aligned}$$

for any $t_1,t_2\in [0,1]$, which proves (3.3).

The rest is obvious. ∎

We can prove now the following Hermite-Hadamard type inequalities for Lipschitzian functions of operators.

Theorem 4. Let $f:\mathcal{C}\subset \mathcal{B}(H)\to \mathcal{B}(H)$ be a vector L-Lipschitzian function on the convex set $\mathcal{C}$. Then we have the inequalities

$$\left\| f\left(\frac{U+V}{2}\right) x-\int_0^1 f\left(\left(1-s\right) U+sV\right) xdt\right\| \le \frac{1}{4}L\left\| Ux-Vx\right\| , \tag{3.7}$$

$$\left\| \frac{f\left(U\right) x+f\left(V\right) x}{2}-\int_0^1 f\left(\left(1-s\right) U+tV\right) xds\right\| \le \frac{1}{4}L\left\| Ux-Vx\right\| \tag{3.8}$$

and

$$\left\| \frac{1}{2}\left[f\left(\frac{3U+V}{2}\right) x+f\left(\frac{U+3V}{2}\right) x\right] -\int_0^1 f\left(\left(1-s\right) U+sV\right) xds\right\| \le \frac{1}{8}L\left\| Ux-Vx\right\| \tag{3.9}$$

for any $U, V \in \mathcal{C}$ and $x \in H$.

Proof. First, observe that $f : \mathcal{C} \subset \mathcal{B}(H) \to \mathcal{B}(H)$ is continuous in the norm topology of $\mathcal{B}(H)$, therefore the integral $\int_0^1 f((1-t)U + tV)\,dt$ exists for any $U, V \in \mathcal{C}$.

Utilising the inequality (3.4) and the norm inequality for norm, we have

$$\begin{aligned}\left\| f\left(\frac{U+V}{2}\right)x - \int_0^1 \varphi_{U,V,x}(t)\,dt \right\| &\le \int_0^1 \left\| f\left(\frac{U+V}{2}\right)x - \varphi_{U,V,x}(t) \right\| dt \\ &\le \frac{1}{2} L \|Ux - Vx\| \int_0^1 (1-t)\,dt \\ &= \frac{1}{4} L \|Ux - Vx\|\end{aligned} \tag{3.10}$$

for any $U, V \in \mathcal{C}$ and $x \in H$.

By the definition of $\varphi_{U,V}$ we have

$$\begin{aligned}&\int_0^1 \varphi_{U,V,x}(t)\,dt \\ &= \frac{1}{2}\left[\int_0^1 f\left((1-t)U + t\frac{U+V}{2}\right) xdt + \int_0^1 f\left(t\frac{U+V}{2} + (1-t)V\right) xdt\right].\end{aligned}$$

Now, using the change of variable $t = 2s$ we have

$$\frac{1}{2}\int_0^1 f\left((1-t)U + t\frac{U+V}{2}\right) xdt = \int_0^{1/2} f((1-s)U + sV)\,xds$$

and by the change of variable $t = 1 - v$ we have

$$\frac{1}{2}\int_0^1 f\left(t\frac{U+V}{2} + (1-t)V\right) xdt = \frac{1}{2}\int_0^1 f\left((1-v)\frac{U+V}{2} + vV\right) xdv.$$

Moreover, if we make the change of variable $v = 2s - 1$ we also have

$$\frac{1}{2}\int_0^1 f\left((1-v)\frac{U+V}{2} + vV\right) xdv = \int_{1/2}^1 f((1-s)U + sV)\,xds.$$

Therefore

$$\begin{aligned}\int_0^1 \varphi_{U,V,x}(t)\,dt &= \int_0^{1/2} f((1-s)U + sV)\,xdt + \int_{1/2}^1 f((1-s)U + sV)\,xds \\ &= \int_0^1 f((1-s)U + sV)\,xdt\end{aligned}$$

and by (3.10) we deduce (3.7).

The other inequalities (3.8) and (3.9) follow in a similar way and the details are omitted. ∎

Corollary 3. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $U,V \in \mathcal{B}(H)$ are commuting and such that $\|U\|, \|V\| \le M < R$, then

$$\left\| f\left(\frac{U+V}{2}\right) x - \int_0^1 f((1-s)U + sV) x ds \right\| \le \frac{1}{4} f_a'(M) \|Ux - Vx\|, \tag{3.11}$$

$$\left\| \frac{f(U)x + f(V)x}{2} - \int_0^1 f((1-s)U + tV) x ds \right\| \le \frac{1}{4} f_a'(M) \|Ux - Vx\| \tag{3.12}$$

and

$$\left\| \frac{1}{2}\left[f\left(\frac{3U+V}{2}\right) x + f\left(\frac{U+3V}{2}\right) x \right] - \int_0^1 f((1-s)U + sV) x ds \right\| \tag{3.13}$$
$$\le \frac{1}{8} f_a'(M) \|Ux - Vx\|,$$

for any $x \in H$.

Proof. Since $U,V \in \mathcal{B}(H)$ are commuting and such that $\|U\|, \|V\| \le M$, then for any $x \in H$ we have by (2.4) that

$$\|f(T)x - f(V)x\| \le f_a'(M) \|Tx - Vx\|.$$

Since the operators $\frac{U+V}{2}$ and $(1-s)U + sV$, $s \in [0,1]$ are commutative, then

$$\left\| f\left(\frac{U+V}{2}\right) x - f((1-s)U + sV) x \right\| \le f_a'(M) \|Tx - Vx\|,$$

and by the argument in Theorem 4 we get (3.11).

The rest can be proved in a similar way and we omit the details. ∎

It is known that if U and V are commuting operators, then the *operator exponential function* $\exp : \mathcal{B}(H) \to \mathcal{B}(H)$ given by

$$\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if A is invertible and $a,b \in \mathbb{R}$ with $a<b$ then

$$\int_a^b \exp(tA)\, dt = A^{-1}[\exp(bA) - \exp(aA)].$$

Proposition 1. Let U and V be commuting operators with $\|U\|, \|V\| \le M$ and such that $V-U$ is invertible. Then we have the inequalities

$$\left\| \exp\left(\frac{U+V}{2}\right) x - (V-U)^{-1}[\exp(V) - \exp(U)] x \right\| \tag{3.14}$$
$$\le \frac{1}{4} \|Ux - Vx\| \exp(M),$$

$$\left\| \frac{\exp(U)x + \exp(V)x}{2} - (V-U)^{-1}[\exp(V) - \exp(U)]x \right\| \leq \frac{1}{4}\|Ux - Vx\| \exp(M) \tag{3.15}$$

and

$$\left\| \frac{1}{2}\left[\exp\left(\frac{3U+V}{2}\right)x + \exp\left(\frac{U+3V}{2}\right)x\right] - (V-U)^{-1}[\exp(V) - \exp(U)]x \right\| \leq \frac{1}{8}\|Ux - Vx\| \exp(M). \tag{3.16}$$

Proof. Follows by Corollary 3 on observing that

$$\begin{aligned}\int_0^1 \exp((1-s)U + sV)\,ds &= \int_0^1 \exp(s(V-U))\exp(U)\,ds \\ &= \left(\int_0^1 \exp(s(V-U))\,ds\right)\exp(U) \\ &= (V-U)^{-1}[\exp(V-U) - I]\exp(U) \\ &= (V-U)^{-1}[\exp(V) - \exp(U)].\end{aligned}$$

∎

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